

# K-THEORY OF TORIC VARIETIES REVISITED

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**ABSTRACT.** After surveying higher  $K$ -theory of toric varieties, we present Totaro's old (c. 1997) unpublished results on expressing the corresponding homotopy theory via singular cohomology. It is a higher analog of the rational Chern character isomorphism for general toric varieties. Apart from its independent interest, in retrospect, Totaro's observations motivated some (old) and complement other (very recent) results. We also offer a conjecture on the nil-groups of affine monoid, extending the nilpotence property. The conjecture holds true for  $K_0$ .

## 1. $K$ -THEORY OF TORIC VARIETIES: SURVEY

**1.1. Conventions.** All our rings and monoids are commutative. Unless specified otherwise, the monoid operation is written additively.

Our monoid and convex geometry terminology follows [3]. In particular, an *affine monoid* is a finitely generated submonoid of a free abelian group. For an affine monoid  $M$  its largest subgroup will be denoted by  $U(M)$ . An affine monoid  $M$  is called (i) *positive* if  $U(M) = 0$ , (ii) *normal* if  $M$  is isomorphic to a monoid of the form  $C \cap \mathbb{Z}^d$  for a *finite rational cone*  $C \subset \mathbb{R}^d$ , and (iii) *seminormal* if, for every element  $x \in \text{gp}(M)$ , the inclusions  $2x, 3x \in M$  imply  $x \in M$ , where  $\text{gp}(M)$  is the group of differences of  $M$ , i. e.,  $\text{gp}(M)$  is the universal group to which  $M$  maps. We say that  $M$  is *simplicial* if the cone  $\mathbb{R}_+ M$  is such

For a functor, defined on rings, a natural number  $c$ , and a ring  $R$  we denote by  $c_*$  the endomorphism, induced by the monoid ring endomorphism  $R[M] \rightarrow R[M]$ ,  $m \mapsto m^c$ , where the monoid operation is written multiplicatively.

For generalities on toric varieties the reader is referred to [3, Chap. 10] and [6].

All fans considered below are assumed to be finite and rational.

Let  $\mathcal{F}$  be a fan. The set of maximal cones in  $\mathcal{F}$  is denoted by  $\max(\mathcal{F})$ . The *toric scheme* over a ring  $R$ , associated with  $\mathcal{F}$ , will be denoted by  $\mathcal{V}_R(\mathcal{F})$ .

For a lattice polytope  $P$ , the *projective toric scheme* over  $R$ , associated with  $P$ , is defined by  $\text{Proj}(R[P]) = \mathcal{V}(\mathcal{N}(P))$ , where  $\mathcal{N}(P)$  is the *normal fan* of  $P$ .

We have  $\mathbb{A}_R^d = \text{Spec}(R[\mathbb{Z}_+^d])$ ,  $\mathbb{T}_R^d = \text{Spec}(R[\mathbb{Z}^d])$ , and  $\mathbb{P}_R^d = \text{Proj}(R[\mathbb{Z}_+^{d+1}]) = \text{Proj}(R[\Delta_d])$ , where  $\Delta_d$  is the unit  $d$ -simplex.

For a monoid  $M$  we put  $M^c = \lim_{\rightarrow} (M \xrightarrow{c_1} M \xrightarrow{c_2} M \xrightarrow{c_3} \dots)$ . Thus, for an abelian group  $G$  and a constant sequence  $\mathbf{c} = (c, c, \dots)$  one has  $G^c = G \otimes \mathbb{Z}[1/c]$ . We will use  $G_{\mathbb{Q}}$  for  $G \otimes \mathbb{Q}$ .

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**1.2. Affine monoid rings.** Let  $R$  be a regular ring. Prototypes for the results discussed below are the classical equalities of Grothendieck  $K_0(R) = K_0(\mathbb{A}_R^d) = K_0(\mathbb{T}_R^d)$  and Berthelot  $K_0(\mathbb{P}_R^d) = K_0(R)^{d+1}$ . Both results were extended to all higher groups due to by Quillen [20]. The following isomorphism of graded rings results from the iterative use of the *fundamental theorem of  $K$ -theory*:

$$(1) \quad K_*(\mathbb{T}_R^d) = K_*(R) \otimes \Lambda^*(\mathbb{Z}^d),$$

where  $\Lambda^1(\mathbb{Z}^d) = \mathbb{Z}^d \subset U(R[\mathbb{Z}^d]) \subset K_1(R[\mathbb{Z}^d])$ .

For affine monoid rings, a substitute for the  $K_*$  homotopy invariance is the following *nilpotence* result. Let  $\mathbf{c} = (c_1, c_2, \dots)$  be a sequence of natural numbers with  $c_i \geq 2$  for all  $i$ ,  $M$  an affine positive monoid, and  $R$  a regular ring. Then

$$(2) \quad \begin{aligned} K_*(R[M^{\mathbf{c}}]) &= K_*(R), \text{ if } \mathbb{Q} \subset R \quad ([13, 15]), \\ K_*(R[M^{\mathbf{c}}]) &= K_*(R), \text{ if } M \text{ is simplicial} \quad ([11, \text{Thm. 6.4}]). \end{aligned}$$

Cortiñas, Haesemeyer, and Weibel [4] gave a shorter and streamlined proof of the first equality (2) when  $R$  is a field of characteristic 0. Both proofs in [4] and [13] use Cortiñas' verification of the *KABI* conjecture [5].

For  $K_1$  and  $K_2$ , the equality (2) can be extended to all regular coefficient rings and all affine monoids. Moreover, the result can be strengthened to the corresponding unstable  $K$ -groups [10, 14].

For the non-positive  $K$ -theory of monoid rings the following stronger equalities hold true. Let  $R$  be a regular ring and  $M$  an affine monoid. Then:

$$(3) \quad \begin{aligned} K_0(R) &= K_0(R[M]) \text{ for } M \text{ seminormal} \quad ([3, \text{Thm. 8.37(a)}]), \\ SK_0(R) &= SK_0(R[M]), \quad ([3, \text{Thm. 8.37(b)}]), \\ K_i(R[M]) &= 0, \quad i < 0. \end{aligned}$$

The triviality of the negative groups in the special case of normal monoids directly follows from the first equality in (3) and the fact that  $K_i(R[M])$  is a direct summand of  $K_0(R[M \times \mathbb{Z}])$  for  $i < 0$ , a consequence of the fundamental theorem. The general case of affine monoids is a little more intricate, but the argument is exactly parallel to the proof of [3, Thm. 8.37(b)].

In the special case  $\text{Krull.dim} R = 1$ , the stronger unstable versions of the first two equalities in (3) are shown in [9, 24].

We will use the following consequence of (1) and (2): for a constant sequence  $\mathbf{c} = (c, c, \dots)$  with  $c \geq 2$ , an affine monoid  $M$ , and a regular ring  $R$ , such that either  $M$  is simplicial or  $\mathbb{Q} \subset R$ ,

$$(4) \quad K_*(R[M^{\mathbf{c}}]) = K_*(R[U(M)^{\mathbf{c}}]) = K_*(R) \otimes \Lambda^*(\mathbb{Z}[1/c]^d).$$

**1.3. Toric varieties.** First we assume  $\mathcal{V} = \mathcal{V}_k(\mathcal{F})$  is a complete and smooth toric variety over a general ground field  $k$  (equivalently,  $\mathcal{F}$  is complete and unimodular). By Grothendieck-Riemann-Roch, Chern character yields the ring homomorphism  $\text{ch} : K_0(\mathcal{V})_{\mathbb{Q}} \rightarrow \text{CH}^*(\mathcal{V})_{\mathbb{Q}}$ . Jurkiewicz and Danilov have long determined the target

Chow ring.<sup>1</sup> As an additive group,  $\mathrm{CH}^*(\mathcal{V}) = \mathbb{Z}^n$ ,  $n = \# \max(\mathcal{F})$ . It is also known that  $K_0(\mathcal{V})$  is a free abelian group (Merkurjev-Panin, Morelli, Vezzosi-Vistoli). As for higher groups, Vezzosi-Vistoli [27] have shown the ring isomorphism

$$(5) \quad K_*(k) \otimes K_0(\mathcal{V}) = K_*(\mathcal{V}).$$

So, additively,  $K_*(\mathcal{V}) = K_*(k)^m$ ,  $m = \# \max(\mathcal{F})$ .

For a general simplicial projective toric variety  $\mathcal{V}$  over a large ground field (such as  $\mathbb{C}$ ),  $K_0(\mathcal{V})$  may contain a nonzero continuous contribution from the higher nil-groups [12] (see also [4]). In Section 3 we offer a conjectural glimpse into the structure of such a continuous parts.

The nilpotence property (2) transfers to general toric varieties as follows. For  $\mathfrak{c} = (c_1, c_2, \dots)$ ,  $c_i \geq 2$ , a fan  $\mathcal{F}$ , and a regular ring  $R$ , such that either  $\mathbb{Q} \subset R$  or  $\mathcal{F}$  simplicial, one has  $K_*(\mathcal{V})^{\mathfrak{c}} = K_*(\mathcal{V} \times \mathbb{A}^1)^{\mathfrak{c}}$  ([11, Prop. 4.7]). Equivalently,  $K_*(\mathcal{V})^{\mathfrak{c}} = KH_*(\mathcal{V})^{\mathfrak{c}}$  ([4, Thm. 6.9]).

**Remark 1.1.** (a) An affine positive monoid  $M$  admits a *grading*, i. e., a partitioning  $M = \{0\} \cup M_1 \cup M_2 \cup \dots$  with  $M_i + M_j \subset M_{i+j}$  ([3, Prop. 2.17]). Consequently, for a field  $k$  of characteristic 0, Strienstra's operations [23] of the big Witt vectors  $W(k)$  on the nil-groups (extended to graded rings in [30]), make  $K_*(k[M])/K_*(k)$  into a  $k$ -vector space. Moreover, the endomorphism  $c_* : K_*(k[M])/K_*(k) \rightarrow K_*(k[M])/K_*(k)$  has no non-zero eigenvalues for  $c \geq 2$ . Totaro uses this observation to conclude that the nil part of the spectral sequence for the  $K$ -theory of a toric variety, associated with the standard open cover, does not interfere with the homotopy  $K$ -theory part. The equalities (2) make the analysis of the spectral sequence straightforward (Section 2.1). Back in the 1990s, Totaro's observation suggested the crucial idea of using the operations to prove the equalities (2) ([13]) – a project, stalled at that time on the classical  $K$ -groups. The full potential of the Strienstra operations in the study of  $K$ -theory of monoid rings does not seem exhausted; see Section 3.

(b) For general projective simplicial toric schemes Corollary 2.3(c) below gives an analog of (5). Namely, for the Weibel homotopy theory one has  $KH_*(\mathcal{V}_R(\mathcal{F}))_{\mathbb{Q}} = K_*(R)_{\mathbb{Q}}^m$ ,  $m = \max(\mathcal{F})$ , where  $R$  is a regular ring and  $\mathcal{F}$  a projective simplicial fan. Recently, Massey [19] has shown that  $KH_*(\mathcal{V}_R(\mathcal{F}))_{\mathbb{Q}}$  depends only on the combinatorial type of  $\mathcal{F}$  for any complete simplicial fan  $\mathcal{F}$ .

Another recent result, due to Hüttemann [16], is that  $K_*(R)^{n_P+1}$  splits off from  $K_*(\mathrm{Proj}(R[P]))$  for any lattice polytope  $P$  and any ring  $R$ , where  $n_P$  be the least non-negative integer for which the dilated polytope  $(n_P + 1)P$  has an interior lattice point. Obviously,  $n_P \leq \dim P$ . It is known that  $n_P$  is the number of distinct integral roots of the *Ehrhart polynomial* of  $P$ . The mentioned splitting recovers the equality  $K_*(\mathbb{P}_R^d) = K_*(R)^{d+1}$  for  $R$  regular. Corollary 2.3 suggest that, for a general lattice polytope  $P$ , the number of copies of  $K_*(R)$ , splitting off from  $K_*(\mathrm{Proj}(R[P]))$  rationally, is at least  $\# \mathrm{vert}(P)$ .

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<sup>1</sup>For the unreferenced results in this subsection see [3, Chap. 8] and the references therein.

## 2. SPECTRAL SEQUENCES, WEIGHTS, AND DEGENERATION

This section is based on Totaro's notes, with the difference that we include more background material, state the results for general regular ground rings, and sometimes use smaller localizations of  $\mathbb{Z}$  than  $\mathbb{Q}$ .

**2.1. Mayer-Vietoris spectral sequence.** Thomason [25, Section 8] established  $K$ -theoretical Mayer-Vietoris long exact sequence for covers by two open subschemes. For covers by more than two open subschemes one obtains strongly convergent spectral sequences [25, Section 8]. The theory, considered in [25], is the Waldhausen theory, associated with the category of perfect complexes. The latter coincides with Quillen's  $K$ -theory for schemes with ample systems of line bundles. The latter include quasi-projective schemes over affine schemes [25, Section 3]. The analogous spectral sequence for  $KH_*$  had been constructed before in [29]. In particular, for a finite open cover  $\mathcal{V} = \bigcup_{i=1}^n U_i$ , one has

$$E_1^{p,q} = \bigoplus_{i_1 < \dots < i_p} k_q(U_{i_1} \cap \dots \cap U_{i_p}) \implies k_{q-p}(\mathcal{V}),$$

where  $k_*$  denotes  $K_*$  or  $KH_*$  and the indexing is that of Bousfield-Kan:

$$d_r : E_r^{p,q} \rightarrow E_r^{p+r, q+r-1}.$$

Let  $\mathcal{F}$  be a fan in  $\mathbb{R}^d$ ,  $\max(\mathcal{F}) = \{\sigma_1, \dots, \sigma_n\}$ , and  $R$  be a regular ring. Denote by  $E$  and  $E'$ , correspondingly, the  $KH_*$ - and  $K_*$ -spectral sequences for the standard open cover of  $\mathcal{V}_R(\mathcal{F})$ .

The affine open subscheme  $U_{\sigma_i}$ , corresponding to  $\sigma_i$ , is of the form

$$\mathrm{Spec}(R[\sigma_i^{\mathrm{op}} \cap (\mathbb{Z}^d)^{\mathrm{op}}]) = \mathrm{Spec}(R[M(\sigma_i) \times M_i])$$

for some affine normal positive monoid  $M_i$ . Here, for a cone  $\sigma \subset \mathbb{R}^d$ , we denote by  $M(\sigma)$  the intersection of the maximal linear subspace of the dual cone  $\sigma^{\mathrm{op}}$  with the dual lattice  $(\mathbb{Z}^d)^{\mathrm{op}}$ . Because  $M_i$  admits a grading, we have  $KH_*(U_{\sigma_i}) = KH_*(R[M(\sigma_i)]) = K_*(R[M(\sigma_i)])$ . So, in view of the third equality in (3), both  $E$  and  $E'$  are first quadrant spectral sequences. By (1), the first page of  $E$  is

$$(6) \quad \dots$$

$$\bigoplus_i \bigoplus_{j=0}^2 (\Lambda^{2-j} M(\sigma_i) \otimes K_j(R)) \longrightarrow \bigoplus_{i_1 < i_2} \bigoplus_{j=0}^2 (\Lambda^{2-j} M(\sigma_{i_1} \cap \sigma_{i_2}) \otimes K_j(R)) \longrightarrow \dots$$

$$\bigoplus_i (M(\sigma_i) \oplus K_1(R)) \longrightarrow \bigoplus_{i_1 < i_2} (M(\sigma_{i_1} \cap \sigma_{i_2}) \oplus K_1(R)) \longrightarrow \dots$$

$$\bigoplus_i \mathbb{Z} \longrightarrow \bigoplus_{i_1 < i_2} \mathbb{Z} \longrightarrow \dots$$

that is,

$$(7) \quad E_1^{p,q} = \bigoplus_{i_1 < \dots < i_p} \bigoplus_{j=0}^q (\Lambda^{q-j} M(\sigma_{i_1} \cap \dots \cap \sigma_{i_p}) \otimes K_j(R)).$$

We also have

$$(E')_1^{p,q} = \left[ \bigoplus_{i_1 < \dots < i_p} \bigoplus_{j=0}^q (\Lambda^{q-j} M(\sigma_{i_1} \cap \dots \cap \sigma_{i_p}) \otimes K_j(R)) \right] \oplus N^{p,q}$$

for certain nil-groups  $N^{p,q}$ .

For any natural number  $c \geq 2$ :

- (i)  $c_*$  acts on  $E$  as well as on  $E'$ ,
- (ii)  $c_*$  acts on  $\Lambda^a M(\sigma_{i_1} \cap \dots \cap \sigma_{i_p})$  by multiplication on  $c^a$ ,
- (iii)  $c_*$  is the identity on  $K_*(R)$ ,

The weights, resulting from the action of  $c_*$  on  $E$  will be referred to as  $\mathbb{N}$ -weights.

If either  $\mathbb{Q} \subset R$  or  $\mathcal{F}$  is a simplicial then, in view of (2), the properties (i-iii) imply that, after tensoring with  $\mathbb{Z}[1/c]$ , there are no non-zero homomorphisms between subquotients of groups in  $E_1$  (showing up in  $E'$ ) and subquotients of groups of the form  $N^{p,q}$ . Since this holds true also for  $c' \geq 2$ , coprime with  $c$ , and  $E'$  maps to  $E$  (due to the natural homomorphism  $K_* \rightarrow KH_*$ ), we conclude that  $E$  and the  $N^{p,q}$  form two spectral subsequences in  $E$  and

$$(8) \quad E' = E \oplus N$$

This splitting, in particular, implies that the map  $K_*(\mathcal{V}_R(\mathcal{F})) \rightarrow KH_*(\mathcal{V}_R(\mathcal{F}))$  is surjective. Actually, by [4, Prop. 5.6], the map  $K_*(\mathcal{V}_R(\mathcal{F})) \rightarrow KH_*(\mathcal{V}_R(\mathcal{F}))$  is always split:

$$(9) \quad K_*(\mathcal{V}_R(\mathcal{F})) = KH_*(\mathcal{V}_R(\mathcal{F})) \oplus N_*.$$

In fact, the corresponding splittings on the open subschemes  $U_{\sigma_{i_1} \cap \dots \cap \sigma_{i_p}} = U_{\sigma_{i_1}} \cap \dots \cap U_{\sigma_{i_p}}$  are compatible. So the splitting transfers to  $\mathcal{V}_R(\mathcal{F})$  because  $K_*$  and  $KH_*$  are the homotopy groups of the homotopy limits of the corresponding Čech cosimplicial spectra ([25, Section 8], [29]).

*Adams weights.* Adams operations in higher  $K$ -theory with expected properties have been defined with various levels of generality by several people. These include operations in the affine and regular cases, or with rational coefficients. Since these works mostly emphasize on Quillen's theory and, simultaneously, utilize the Mayer-Vietoris property, one usually requires the existence of an ample family of line bundles (e. g., the quasi-projectivity condition) for the scheme in question. (It seems one can define operations for Thomason's theory in the appropriate generality.)

For a commutative ring  $R$ , Adams operations  $\psi^k : K_*(R) \rightarrow K_*(R)$ ,  $k \neq 0$ , are ring homomorphisms. Let  $R$  be a Noetherian ring of Krull dimension  $d$  and  $q \in \mathbb{Z}_+$ . Denote by  $K_q(R)^{(i)} \subset K_q(R)$  the subgroup of elements of *Adams weight*  $i$ , i. e., the

elements  $x$  such that  $\psi^k(x) = k^i x$  for all  $k$ . Then one has:

$$(10) \quad \begin{aligned} K_q(R) \otimes \mathbb{Z}[1/(q+d-1)!] &= \bigoplus_{i=2}^{q+d} (K_q(R)^{(i)} \otimes \mathbb{Z}[1/(q+d-1)!]) \quad \text{for } q \geq 2, \\ K_1(R) \otimes \mathbb{Z}[1/(d+1)!] &= \bigoplus_{i=1}^{d+1} (K_1(R)^{(i)} \otimes \mathbb{Z}[1/(d+1)!]), \\ U(R) &= U(R)^{(1)}, \quad (\text{i. e., } \psi^k(u) = ku \text{ for all } u \in U(R)) \end{aligned}$$

The first two equalities are proved in [22, §2.8] and the third in [17, Cor. 6.8].

*Notice.* The original formulation in [22] uses the *stable range* of  $R$ , but the letter is known to be bounded above by  $d+1$  [2, Ch. 5, Thm. 3.5].

We also need that, for a quasiprojective scheme and an open cover, the differentials in the Mayer-Vietoris spectral sequence, tensored with  $\mathbb{Q}$ , commute with the Adams operations. This follows from the functoriality of Adams operations with rational coefficients [18]. In fact, since the operations are defined in terms of  $K$ -theoretical spaces, they are natural w.r.t. to the groups of the homotopy fibers, induced by morphisms of the underlying schemes. So Adams operation are natural w.r.t. Mayer-Vietoris long exact sequences, and the naturality can be promoted to Mayer-Vietoris spectral sequences along the lines of proof [25, Prop. 8.3].

**2.2. Singular cohomology.** In this subsection the ground ring is  $\mathbb{C}$ . For any finitely generated graded  $\mathbb{C}$ -algebra  $B = A \oplus A_1 \oplus A_2 \oplus \dots$  the one-parameter family of maps  $B \rightarrow B$ ,  $(a, a_1, a_2, \dots) \rightarrow (a, ta_1, t^2 a_2, \dots)$ ,  $t \in [0, 1]$ , shows that  $\text{Spec}(A)_{\mathbb{C}}$  is a strong deformation retract of  $\text{Spec}(B)_{\mathbb{C}}$ . Any affine toric variety is of the form  $\mathcal{V} = \text{Spec}(\mathbb{C}[M \times \mathbb{Z}^n])$  for an affine normal positive monoid  $M$ . Because there is a grading  $\mathbb{C}[M \times \mathbb{Z}^n] = \mathbb{C}[\mathbb{Z}^n] \oplus A_1 \oplus \dots$ , the space  $\mathcal{V}_{\mathbb{C}}$  is homotopic to the complex torus  $\mathbb{T}_{\mathbb{C}}^n$ , which is in turn homotopic to the real torus  $(S^1)^n$ .

Let  $\mathcal{F}$  be a fan,  $\max(\mathcal{F}) = \{\sigma_1, \dots, \sigma_n\}$ ,  $\mathcal{V} = \mathcal{V}_{\mathbb{C}}(\mathcal{F})$ . Let  $M(\sigma)$  and  $U_{\sigma}$  be as in Section 2.1.

Because  $U_{\sigma_1 \cap \dots \cap \sigma_p}$  is homotopic to  $(S^1)^r$  with  $r = \text{rank } M(\sigma_1 \cap \dots \cap \sigma_p)$ , we have  $H^q(U_{\sigma_1 \cap \dots \cap \sigma_p}, \mathbb{Z}) = \Lambda^q M(\sigma_1 \cap \dots \cap \sigma_p)$ . Correspondingly, the first page of the integral cohomology spectral sequence  $\tilde{E}$  of the standard open cover  $\mathcal{V} = \bigcup U_{\sigma_i}$  is

$$(11) \quad \begin{aligned} \oplus_i \Lambda^2 M(\sigma_i) &\longrightarrow \oplus_{i_1 < i_2} \Lambda^2 M(\sigma_{i_1} \cap \sigma_{i_2}) \longrightarrow \oplus_{i_1 < i_2 < i_3} \Lambda^2 M(\sigma_{i_1} \cap \sigma_{i_2} \cap \sigma_{i_3}) \longrightarrow \dots \\ \oplus_i M(\sigma_i) &\longrightarrow \oplus_{i_1 < i_2} M(\sigma_{i_1} \cap \sigma_{i_2}) \longrightarrow \oplus_{i_1 < i_2 < i_3} M(\sigma_{i_1} \cap \sigma_{i_2} \cap \sigma_{i_3}) \longrightarrow \dots \\ \oplus_i \mathbb{Z} &\longrightarrow \oplus_{i_1 < i_2} \mathbb{Z} \longrightarrow \oplus_{i_1 < i_2 < i_3} \mathbb{Z} \longrightarrow \dots \end{aligned}$$

where the differentials are  $d^r : \tilde{E}_r^{p,q} \rightarrow \tilde{E}_r^{p+r,q-r+1}$ , i. e.,

$$\tilde{E}_1^{p,q} = \oplus_{i_1 < \dots < i_p} \Lambda^q M(\sigma_{i_1} \cap \dots \cap \sigma_{i_p}) \implies H^{p+q}(\mathcal{V}, \mathbb{Z})$$

*Notice.* In practice, it is difficult to keep track of the many cones in  $\mathcal{F}$ . To circumvent the difficulty, one usually uses another more economy spectral sequences – the spectral sequence resulting from filtering  $\mathcal{V}$  by the unions of the closures of the torus orbits [6, Chap. 12].

As usual, the action of a natural number  $c$  on the spectral sequence  $\tilde{E}$  will be denoted by  $c_*$ . We see that the rational cohomology  $H^q(\mathcal{V}, \mathbb{Q})$  is filtered

$$0 \subset V_0 \subset \dots \subset V_q = H^q(\mathcal{V}, \mathbb{Q})$$

in such a way that  $c_*$  acts on  $V_j/V_{j-1}$  by multiplication on  $c^j$ .

When  $\mathcal{V}$  is complete (i. e., when  $\mathcal{F}$  is a complete fan), the corresponding filtration on  $H^q(\mathcal{V}, \mathbb{Q})$  coincides with the *Deligne weight filtration*. More precisely, one has ([1, Prop. 1.4][28]):

$$\begin{aligned} W_0 H^q(\mathcal{V}, \mathbb{Q}) &\subset W_1 H^q(\mathcal{V}, \mathbb{Q}) \subset \dots \\ &\subset W_{2q-1} H^q(\mathcal{V}, \mathbb{Q}) \subset W_{2q} H^q(\mathcal{V}, \mathbb{Q}) = H^q(\mathcal{V}, \mathbb{Q}), \end{aligned}$$

where, as a special feature of toric varieties,

$$\begin{aligned} W_{2j} H^q(\mathcal{V}, \mathbb{Q}) &= W_{2j+1} H^q(\mathcal{V}, \mathbb{Q}) = \{z \in H^q(\mathcal{V}, \mathbb{Q}) \mid c_*(Z) = c^i z \text{ for some } i \leq j\}, \\ &j = 0, \dots, q-1. \end{aligned}$$

(For noncompact varieties, Deligne filtration is defined for the rational cohomology with compact support [7], but we do not need it here.)

Under the same completeness condition Totaro [26, Thm. 3] has shown

$$(12) \quad \bigoplus_{a \geq 0} \mathrm{gr}_{2a}^W H^{2a}(\mathcal{V}, \mathbb{Q}) = A^*(\mathcal{V})_{\mathbb{Q}},$$

where  $W$  refers to the  $\mathbb{N}$ -weights with doubled indices (to make it compatible with Deligne's weights in the complete case) and  $A^*$  is the operational Chow cohomology. In fact, [26, Thm. 3] establishes the dual rational isomorphism between the Chow groups and the smallest subspace of the Borel-Moore homology with respect to the weight filtration, the duality between  $A_*$  and  $A^*$  being provided by [8, Thm. 3].

**2.3. Degeneration at the second page.** The following is a standard degeneration result, usually presented for the rational cohomology (e. g., [6, Prop. 12.3.10]).

**Lemma 2.1.** *For any integer  $c \geq 2$ , the spectral sequence in Section 2.2 degenerates at  $\tilde{E}_2$  after tensoring with  $\mathbb{Z}[1/c]$ .*

*Proof.* For any  $r$ , the group  $\tilde{E}_r^{p,q}$  is a subquotient of  $\tilde{E}_1^{p,q}$ . Therefore,  $c_*$  acts on  $\tilde{E}_r^{p,q}$  by multiplication on  $c^q$ . Let  $z \in \tilde{E}_r^{p,q}$ . Then  $d_r(z) \in \tilde{E}_r^{p+r, q-r+1}$  and we have

$$c^{q-r+1} d_r^{p,q}(z) = c_*(d_r^{p,q}(z)) = d_r^{p,q}(c_*(z)) = d_r^{p,q}(c^q z) = c^q d_r^{p,q}(z),$$

implying  $d_r^{p,q}(z) = 0$  for  $r \geq 2$ . □

Next lemma is Totaro's degeneration result for the spectral sequence  $E$  in Section 2.1. The argument is a variation of the proof of Lemma 2.1. The points is to use the two different splitting, provided by the  $\mathbb{N}$ - and Adams weights.

**Lemma 2.2.** *Let  $R$  be a regular ring and  $\mathcal{F}$  a quasi-projective fan. Assume either  $\mathbb{Q} \subset R$  or  $\mathcal{F}$  is simplicial. Then  $E_{\mathbb{Q}}$  degenerates at the second page.*

*Proof.* Let  $\sigma_i, U_{\sigma_i}, M(\sigma)$  be as in Section 2.1. By (8) and the end remark in Section 2.1, the differentials in  $E$  commute with Adams operations.

We can assume  $n \geq 2$ , for otherwise there is nothing to prove.

First, we observe that the differential  $d_1$  is an integer linear combination of restriction maps  $K_*(U) \rightarrow K_*(U')$  for  $U' \subset U$ . In particular, the subgroup  $\Lambda^* M(\sigma) \subset K_1(U) \subset K_*(U)$  restricts to the subgroup  $\Lambda^* M(\sigma') \subset K_1(U') \subset K_*(U')$ . Consequently, if we let  $A$  be the part of the first page, consisting of the groups of type  $\Lambda^* M$  only, then  $d_1$  maps  $A$  into itself. Because  $E$  is a module over  $K_*(R)$  and the first page is  $A \otimes K_*(R)$ , the second page is  $H(A, d) \otimes K_*(R)$ . So to show that the spectral sequence, tensored with  $\mathbb{Q}$ , degenerates at the second page, it suffices to show that all differentials starting at  $H(A, d)_{\mathbb{Q}}$  are 0.

The subgroup of the first page of  $\mathbb{N}$ -weight  $c$  has the form  $\Lambda^c M \otimes K_{q-c}(R)$  for some lattice  $M$ . By (10), for  $q > 0$  we have: (i) the group  $K_q(R)_{\mathbb{Q}}$  splits into subgroups of positive Adams weights and (ii) the subgroup  $M_{\mathbb{Q}} \subset K_1(U)_{\mathbb{Q}}$  (for an appropriate  $U$ ) has Adams weight 1. Since the maps  $\psi^k$  respect the multiplicative structure of  $K_*$ , the group  $\Lambda^c M \otimes K_{q-c}(R)$  for  $c < q$  has Adams weight  $> c$ .

Consider a differential  $d_r : (\text{subquotient of } \Lambda^q M_{\mathbb{Q}}) \rightarrow (E_{\mathbb{Q}})_r^{p+r, q+r-1}$ . The  $\mathbb{N}$ - and Adams weights of the source are both  $q$ . But for  $r \geq 2$ , the previous paragraph shows that the part of the target group of  $\mathbb{N}$ -weight  $q$  has the form  $\Lambda^q M'_{\mathbb{Q}} \otimes K_{r-1}(R)_{\mathbb{Q}}$  for some lattice  $M'$ . By the same paragraph, the latter group splits into subgroups of Adams weight  $> q$ . Since the weights must be respected, we conclude  $d_r = 0$ .  $\square$

*Notice.* If the Adams operations on the affine charts of  $\mathcal{V}$  commute with the differentials in  $E$  without inverting the integers, then (10) and the argument above imply the stronger degeneration  $E_{p,q}^2 \otimes \mathbb{Z}[1/c!] = E_{p,q}^3 \otimes \mathbb{Z}[1/c!] = \cdots = E_{p,q}^{\infty} \otimes \mathbb{Z}[1/c!]$  where  $d = \text{Krull.dim}(R)$  and  $c \geq n + q + d - 1$ .

**Corollary 2.3.** *Let  $\mathcal{F}$  be a quasi-projective fan,  $R$  a regular ring, and  $\mathcal{V} = \mathcal{V}_{\mathbb{C}}(\mathcal{F})$ . Assume either  $\mathbb{Q} \subset R$  or  $\mathcal{F}$  is simplicial.*

(a) *The map  $K_*(\mathcal{V}_R(\mathcal{F})) \rightarrow KH_*(\mathcal{V}_R(\mathcal{F}))$  is split surjective and*

$$KH_*(\mathcal{V}_R(\mathcal{F}))_{\mathbb{Q}} = (\text{gr}_*^W H^*(\mathcal{V}, \mathbb{Q})) \otimes K_*(R).$$

*More precisely,*

$$KH_n(\mathcal{V}_R(\mathcal{F}))_{\mathbb{Q}} = \left[ \bigoplus_{p,q \geq 0} \text{gr}_{2(p+n-q)}^W H^{2p+n-q}(\mathcal{V}, \mathbb{Q}) \right] \otimes K_q(R).$$



(b) If  $\mathcal{F}$  is complete then

$$KH_0(\mathcal{V}_R(\mathcal{F}))_{\mathbb{Q}} = \left[ \bigoplus_{p \geq 0} \text{gr}_{2p}^W H^{2p}(\mathcal{V}, \mathbb{Q}) \right] \otimes K_0(R) = A^*(\mathcal{V})_{\mathbb{Q}} \otimes K_0(R).$$

(c) If  $\mathcal{F}$  is projective simplicial then, additively,

$$KH_*(\mathcal{V}_R(\mathcal{F}))_{\mathbb{Q}} = K_*(R)_{\mathbb{Q}}^m, \quad m = \max(\mathcal{F}).$$

*Proof.* (a) The splitting result is given in (9).

By Lemma 2.2,  $KH_n(\mathcal{V}_R(\mathcal{F}))_{\mathbb{Q}}$  is the direct sum of the graded rational pieces which align in  $(E_{\mathbb{Q}})_2^{p,q}$  along the shifted diagonal  $y = x + n$ . Since the homomorphisms in (7) are the same as in (11) with certain shift (and tensored with the  $K_j(R)$ ), we only need to keep track of the graded pieces from  $H^*(\mathcal{V}, \mathbb{Q})$ , picked up in the mentioned summation process. This is most conveniently done by groupings w.r.t to the tensor factors  $K_q(R)$ .

(b) This follows from (a) and (12).

(c) By Oda's theorem [6, Thm. 12.3.11],  $H^{2p}(\mathcal{V}, \mathbb{Q})$  is *pure*, i. e.,  $H^{2p}(\mathcal{V}, \mathbb{Q}) = \text{gr}_{2p}^W H^{2p}(\mathcal{V}, \mathbb{Q})$ . (As remarked above, the cohomology spectral sequence, used in [6, Ch. 12], is different from  $\tilde{E}$ .) On the other hand, we have the following Betti number counting ([6, Thm. 12.3.12])

$$b_{2p}(\mathcal{V}) = \dim H^{2p}(\mathcal{V}, \mathbb{Q}) = \sum_{i=p}^{\dim \mathcal{F}} (-1)^{i-p} \binom{i}{p} \# \mathcal{F}(d-i), \quad p \geq 0,$$

where  $\mathcal{F}(d-i)$  is the set of  $(d-i)$ -dimensional cones in  $\mathcal{F}$ . So by (a) we have  $KH_*(\mathcal{V}_R(\mathcal{F}))_{\mathbb{Q}} = K_*(R)_{\mathbb{Q}}^{b_0(\mathcal{V})+b_2(\mathcal{V})+b_4(\mathcal{V})+\dots} = K_*(R)_{\mathbb{Q}}^m$ .  $\square$

*Notice.* Most likely, Corollary 2.3(c) extends to all complete simplicial fans  $\mathcal{F}$ : word-by-word the same argument goes through if Adams operation commute with the corresponding Mayer-Vietoris differentials.

### 3. NIL-GROUPS

For an affine monoid  $M$  the *normalization* and *seminormalization* of  $M$  are defined as follows

$$\begin{aligned} \text{n}(M) &= \{x \in \text{gp}(M) \mid cx \in M \text{ for some } c \in \mathbb{N}\}, \\ \text{sn}(M) &= \{x \in \text{gp}(M) \mid cx \in M \text{ for all sufficiently large } c \in \mathbb{N}\}. \end{aligned}$$

They are, correspondingly, the smallest seminormal and normal monoids in  $\text{gp}(M)$ , containing  $M$ .

For a cone  $C$  we denote by  $\mathbf{F}(C)$  the set of nonzero faces of  $C$  and by  $\text{int}(C)$  the relative interior of  $C$  in the ambient Euclidean space.

**Conjecture 3.1.** Let  $n$  be a nonnegative integer,  $M$  a positive affine monoid,  $R$  a regular ring with  $\mathbb{Q} \subset R$ . Denote  $\tilde{K}_n(R[M]) = K_n R[M] / K_n(R)$ . Then there is a  $\text{gp}(M)$ -grading

$$\tilde{K}_n(R[M]) = \bigoplus_{\text{sn}(M)} A_j,$$

satisfying the conditions:

- (a) For every  $F \in \mathbf{F}(C)$  there exist  $m_F \in \text{int}(F) \cap \text{sn}(M)$  such that

$$\text{Supp}(\tilde{K}_n(R[M])) \subset \text{sn}(M) \setminus \bigcup_{\mathbf{F}(C)} (m_F + F);$$

- (b)  $\text{Supp}(c_*(x)) \subset c \text{Supp}(x)$  for all  $x \in \tilde{K}_n(R[M])$  and  $c \geq 2$ ;  
(c) There is an  $R[M]$ -module structure on  $\tilde{K}_n(R[M])$  with  $c_*(mx) = m^c c_*(x)$ ,  
(d) As an  $R[M]$ -module,  $\tilde{K}_n(R[M])$  is generated by  $\bigoplus_J A_j$  for a finite subset  $J \subset \text{sn}(M)$ . Moreover, when  $R$  is a number field,  $\tilde{K}_i(R[M])$  is a finitely generated  $R[M]$ -module.

The conjecture is true for  $i = 0$ . In fact, by [3, Thm. 8.42],  $\text{Pic}(R[M]) / \text{Pic}(R) = R(\text{sn}(M) \setminus M)$ , the free  $R$  module on  $\text{sn}(M) \setminus M$ . Actually, this is the monoid-ring case of [21]: the latter proves the same equality for Picard groups of graded rings. However, the ‘monoid friendly’ argument, presented in [3], makes clear that the  $c_*$ -action on  $\text{Pic}(R[M]) / \text{Pic}(R)$  is exactly the  $c$ -homothety on  $R(\text{sn}(M) \setminus M)$ . Moreover, the second equality in (3) implies  $\tilde{K}_0(R[M]) = \text{Pic}(R[M]) / \text{Pic}(R)$ . Further, the  $R$ -module structure is a consequence of the Strienstra  $W(R)$ -action (upon fixing a grading on  $R[M]$ ) and the fact that  $R \subset W(R)$  because  $\mathbb{Q} \subset R$ . This shows (b,c,d), and (a) follows from the fact that for any affine positive monoid  $L$  there exists  $l \in \text{int}(\mathbb{R}_+ L) \cap L$  such that  $l + n(L) \subset L$  [3, Prop. 2.33].

*Notice.* The starting point here could have been a multivariate version of [23]. Observe, Conjecture 3.1(a) implies the existence of a natural number  $j = j(R, M, n)$  such that the  $j$ -fold iteration of  $c_*$  annihilates all of  $\tilde{K}_n(R[M])$ .

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